Chapter 4 Appendix 3 Extra terms

The Hyperbolic method has been used to prove that

$$\sum_{n\le x} d(n) = x\log x + (2\gamma - 1)\,x + O\big(x^{1/2}\big)\,. \tag{1}$$

This can be used within the Convolution Method to prove

Theorem 1 *There exists a constant C_1 such that*

$$\sum_{n\le X} 2^{\omega(n)} = \frac{1}{\zeta(2)} X\log X + C_1 X + O\big(X^{1/2}\log X\big)\,.$$

Solution Recall that $2^\omega = d * \mu_2$ so

$$\begin{aligned}\sum_{n\le X} 2^{\omega(n)} &= \sum_{a\le X}\mu_2(a)\sum_{b\le X/a} d(b)\\ &= \sum_{a\le X}\mu_2(a)\left(\frac{X}{a}\log\frac{X}{a} + (2\gamma-1)\,\frac{X}{a} + O\left(\left(\frac{X}{a}\right)^{1/2}\right)\right),\end{aligned}$$

by (1). Recall that $\mu_2(a)$ is non-zero only if $a = m^2$, say. Thus

$$\sum_{n\le X} 2^{\omega(n)} = \sum_{m^2\le X}\mu(m)\left(\frac{X}{m^2}\log\frac{X}{m^2} + (2\gamma-1)\,\frac{X}{m^2} + O\left(\left(\frac{X}{m^2}\right)^{1/2}\right)\right). \tag{2}$$

The error here is

$$O\left(X^{1/2}\sum_{m\le X^{1/2}}\frac{1}{m}\right) = O\big(X^{1/2}\log X\big)\,.$$

For the first term in (2) we use a trick of writing the logarithm as an integral and then interchanging it with the summation:

$$\sum_{m^2\le X}\mu(m)\,\frac{X}{m^2}\log\frac{X}{m^2} = X\sum_{m^2\le X}\frac{\mu(m)}{m^2}\int_{m^2}^{X}\frac{dt}{t} = X\int_1^X\sum_{m^2\le t}\frac{\mu(m)}{m^2}\frac{dt}{t}. \tag{3}$$

In this integrand the sum converges absolutely so we complete it to infinity,

$$\sum_{m^2\le t}\frac{\mu(m)}{m^2} = \sum_{m=1}^{\infty}\frac{\mu(m)}{m^2}-\sum_{m>t^{1/2}}\frac{\mu(m)}{m^2} = \frac{1}{\zeta(2)}+O\left(\sum_{m>t^{1/2}}\frac{1}{m^2}\right)$$

$$= \frac{1}{\zeta(2)}+\varepsilon(t)\,, \tag{4}$$

where $\varepsilon(t)=O\big(1/t^{1/2}\big)$. Inserted into (3) this gives

$$\sum_{m^2\le X}\mu(m)\,\frac{X}{m^2}\log\frac{X}{m^2} = X\int_1^X\left(\frac{1}{\zeta(2)}+\varepsilon(t)\right)\frac{dt}{t} = \frac{X\log X}{\zeta\,(2)}+X\int_1^X\varepsilon(t)\,\frac{dt}{t}.$$

Since $\varepsilon(t)=O\big(1/t^{1/2}\big)$ the integral

$$\int_1^{\infty}\varepsilon(t)\,\frac{dt}{t}$$

converges and so is a constant we will denote by C_0. Then

$$\int_1^X\varepsilon\,(t)\,\frac{dt}{t} = C_0-\int_X^{\infty}\varepsilon\,(t)\,\frac{dt}{t} = C_0+O\bigg(\int_X^{\infty}\frac{dt}{t^{3/2}}\bigg) = C_0+O\bigg(\frac{1}{X^{1/2}}\bigg)\,.$$

Hence

$$\sum_{m^2\le X}\mu(m)\,\frac{X}{m^2}\log\frac{X}{m^2} = \frac{X\log X}{\zeta(2)}+C_0X+O\big(X^{1/2}\big)\,.$$

The result (4) with $t=X^{1/2}$ will deal with the remaining term in (2) to give

$$\sum_{n\le X}2^{\omega(n)} = \frac{1}{\zeta(2)}X\log X+C_0X+O\big(X^{1/2}\big)+$$

$$+\,(2\gamma-1)\,X\left(\frac{1}{\zeta(2)}+O\bigg(\frac{1}{X^{1/2}}\bigg)\right)$$

$$+O\big(X^{1/2}\log X\big)\,.$$

This gives the stated result with

$$C_1=C_0+\frac{1}{\zeta(2)}\,(2\gamma-1)\,.$$

■

Question Generalise the previous result and prove that there exists a constant D_k such that

$$\sum_{n\le x} d*\mu_k(n) = \frac{1}{\zeta(k)} x\log x + D_k x + O\big(x^{1/2}\big)\,.$$

for $k \ge 3$. (So there is no log term in the error when $k \ge 3$).

Solution Start as above

$$\begin{aligned}
\sum_{n\le x} d*\mu_k(n) &= \sum_{a\le x}\mu_k(a)\sum_{b\le x/a} d(b)\\
&= \sum_{a\le x}\mu_k(a)\left(\frac{x}{a}\log\frac{x}{a} + (2\gamma-1)\,\frac{x}{a} + O\bigg(\Big(\frac{x}{a}\Big)^{1/2}\bigg)\right) \qquad (5)\\
&= \sum_{m^k\le x}\mu(m)\left(\frac{x}{m^k}\log\frac{x}{m^k} + (2\gamma-1)\,\frac{x}{m^k} + O\bigg(\Big(\frac{x}{m^k}\Big)^{1/2}\bigg)\right).
\end{aligned}$$

For the first term we proceed as

$$\begin{aligned}
\sum_{m^k\le x}\mu(m)\,\frac{x}{m^k}\log\frac{x}{m^k} &= \int_1^x\left(\frac{1}{\zeta(k)} + \varepsilon_k(t)\right)\frac{dt}{t}\\
&= \frac{1}{\zeta(k)}\log x + \int_1^x \varepsilon_k(t)\,\frac{dt}{t},
\end{aligned}$$

where $\varepsilon_k(t) \ll 1/t^{1-1/k}$. Because the integral converges it can be completed to

$$C_k = \int_1^\infty \varepsilon_k(t)\,\frac{dt}{t},$$

say, with an error

$$\le \int_x^\infty \varepsilon_k(t)\,\frac{dt}{t} \ll \int_x^\infty \frac{dt}{t^{2-1/k}} \ll \frac{1}{x^{1-1/k}}.$$

Hence

$$\sum_{m^k\le x}\mu(m)\,\frac{x}{m^k}\log\frac{x}{m^k} = \frac{1}{\zeta(k)}\log x + C_k + O\bigg(\frac{1}{x^{1-1/k}}\bigg),$$

The second term in (5) is

$$(2\gamma-1)\,x \sum_{m\le x^{1/k}} \frac{\mu(m)}{m^k} = (2\gamma-1)\,x\left(\frac{1}{\zeta(k)} + O\left(\frac{1}{\left(x^{1/k}\right)^{k-1}}\right)\right).$$

While the error in (5) is

$$O\left(x^{1/2}\sum_{m\le x^{1/k}} \frac{1}{m^{k/2}}\right).$$

The sum here converges since $k \ge 3$, and so is bounded by a constant. Combining these three results we get the stated result with

$$D_k = C_k + \frac{(2\gamma-1)}{\zeta(k)}.$$

Additional terms for $\sum_{n\le x} d\left(n^2\right)$.

Let $g(n) = d(n^2)$ where d is the divisor function. In Problem Sheet 2 we have $g = 1 * 1 * 1 * \mu_2 = d * Q_2$. Can the Hyperbolic Method be used to improve previous results on $\sum_{n\le x} d(n^2)$?

To do so need to first prove two lemmas

Lemma 2 *There exists a constant C_d say for which*

$$\sum_{a\le U} \frac{d(a)}{a} = \frac{1}{2}\log^2 U + 2\gamma \log U + C_d + O\left(\frac{1}{U^{1/2}}\right).$$

Proof By Partial Summation and (1),

$$\begin{aligned}\sum_{a\le U} \frac{d(a)}{a} &= \frac{1}{U}\sum_{a\le U} d(a) + \int_1^U \sum_{a\le t} d(a)\,\frac{dt}{t^2} \\ &= \frac{1}{U}\left(U\log U + (2\gamma-1)\,U + O\left(U^{1/2}\right)\right) \\ &\qquad + \int_1^U \left(t\log t + (2\gamma-1)\,t + \varepsilon(t)\right)\frac{dt}{t^2}\end{aligned}$$

where $\varepsilon(t) \ll t^{1/2}$. This gives the stated result with

$$C_d = 2\gamma - 1 + \int_1^\infty \varepsilon(t)\,\frac{dt}{t^2}.$$

■

Lemma 3 *There exists a constant C_Q say, for which*

$$\sum_{b\le V} \frac{Q_2(b)}{b} = \frac{1}{\zeta(2)}\log V + C_Q + O\left(\frac{1}{V^{1/2}}\right).$$

Proof By Partial Summation and the sums of square-free numbers

$$\begin{aligned}\sum_{b\le V}\frac{Q_2(b)}{b} &= \frac{1}{V}\sum_{b\le V}Q_2(b)+\int_1^V\sum_{b\le t}Q_2(b)\,\frac{dt}{t^2}\\ &= \frac{1}{V}\left(\frac{1}{\zeta(2)}V+O\big(V^{1/2}\big)\right)\\ &\qquad +\int_1^V\left(\frac{1}{\zeta(2)}t+\varepsilon(t)\right)\frac{dt}{t^2}\end{aligned}$$

where $\varepsilon(t)\ll t^{1/2}$. This gives the stated result with

$$C_d=\frac{1}{\zeta\,(2)}+\int_1^\infty\varepsilon(t)\,\frac{dt}{t^2}.$$

■

Theorem 4 *There exist constants c_1 and c_2 such that*

$$\sum_{n\le x}d\big(n^2\big)=\frac{1}{2\zeta\,(2)}x\log^2x+c_1x\log x+c_2x+O\big(x^{3/4}\log x\big)$$

Proof With U and V to be chosen, the Hyperbolic Method gives

$$\begin{aligned}\sum_{n\le X}d\big(n^2\big)=\sum_{n\le X}d*Q_2(n) &= \sum_{a\le U}d(a)\sum_{b\le X/a}Q_2(b)+\sum_{b\le V}Q_2(b)\sum_{a\le X/b}d(a)\\ &\quad -\left(\sum_{a\le U}d(a)\right)\left(\sum_{b\le V}Q_2(b)\right).\end{aligned}$$

By the Lemmas above,

$$\sum_{a\le U}d(a)\sum_{b\le X/a}Q_2(b)=\sum_{a\le U}d(a)\left(\frac{1}{\zeta(2)}\frac{x}{a}+O\left(\left(\frac{x}{a}\right)^{1/2}\right)\right),\tag{6}$$

and

$$\sum_{b\le V}Q_2(b)\sum_{a\le X/b}d(a)=\sum_{b\le V}Q_2(b)\left(\frac{x}{b}\log\frac{x}{b}+(2\gamma-1)\,\frac{x}{b}+O\left(\left(\frac{x}{b}\right)^{1/2}\right)\right).\tag{7}$$

Errors

The first error, in (6) is, by partial summation

$$\begin{aligned} x^{1/2}\sum_{a\le U}\frac{d(a)}{a^{1/2}} &= x^{1/2}\left(\frac{1}{U^{1/2}}\sum_{a\le U}d(a)+\frac{1}{2}\int_1^U\sum_{a\le t}d(a)\,\frac{dt}{t^{3/2}}\right)\\ &\ll x^{1/2}\left(\frac{1}{U^{1/2}}U\log U+\frac{1}{2}\int_1^U t\log t\frac{dt}{t^{3/2}}\right)\\ &\ll x^{1/2}U^{1/2}\log U. \end{aligned}$$

The second error, in (7) is, again by partial summation,

$$\begin{aligned} x^{1/2}\sum_{b\le V}\frac{Q_2(b)}{b^{1/2}} &= x^{1/2}\left(\frac{1}{V^{1/2}}\sum_{b\le V}Q_2(b)+\frac{1}{2}\int_1^V\sum_{b\le t}Q_2(b)\,\frac{dt}{t^{3/2}}\right)\\ &\ll x^{1/2}\left(\frac{1}{V^{1/2}}V+\frac{1}{2}\int_1^V t\frac{dt}{t^{3/2}}\right)\\ &\ll x^{1/2}V^{1/2} \end{aligned}$$

The idea would be to equate these errors, which means demanding $U = V$. Since $UV = x$ this means that $U = V = x^{1/2}$, when the two errors above are $\ll x^{3/4}\log x$.

An error will also arise from the first term in (6) ,

$$\frac{x}{\zeta(2)}\sum_{a\le U}\frac{d(a)}{a}=\frac{x}{\zeta(2)}\left(\frac{1}{2}\log^2 U+2\gamma\log U+C_d\right)+O\Big(\frac{x}{U^{1/2}}\Big),$$

but this is only $O\left(x^{3/4}\right)$ with our choice of $U = x^{1/2}$.

In the first term in (7) we cannot immediately use the idea of writing the logarithm as an integral. This is because it is the logarithm of x/b where as the sum is over $b \le V$ and not $b \le x$. So first write

$$\log\frac{x}{b}=\log\frac{x}{V}+\log\frac{V}{b}.$$

The first term is independent of b and can be taken out of the sum. The second term has the factor V/b and the sum is over $b \le V$, thus the idea of writing this term as an integral will work.

$$\begin{aligned}
x\sum_{b\le V}\frac{Q_2(b)}{b}\log\frac{x}{b} &= x\sum_{b\le V}\frac{Q_2(b)}{b}\left(\log\frac{x}{V}+\log\frac{V}{b}\right)\\
&= x\log\frac{x}{V}\sum_{b\le V}\frac{Q_2(b)}{b}+x\int_1^V\sum_{b\le t}\frac{Q_2(b)}{b}\frac{dt}{t}\\
&= x\log\frac{x}{V}\left(\frac{1}{\zeta(2)}\log V+C_Q+O\Big(\frac{1}{V^{1/2}}\Big)\right) \qquad (8)\\
&\quad +x\int_1^V\left(\frac{1}{\zeta(2)}\log t+C_Q+\eta(t)\right)\frac{dt}{t}
\end{aligned}$$

where $\eta(t)\ll 1/t^{1/2}$. The integral here equals

$$\frac{1}{2\zeta(2)}\log^2 V+C_Q\log V+D+O\Big(\frac{1}{V^{1/2}}\Big), \qquad (9)$$

where

$$D=\int_1^\infty \eta(t)\frac{dt}{t}.$$

The errors in (8) and (9) are $O\big(x(\log x/V)/V^{1/2}\big)$ and $O\big(x/V^{1/2}\big)$ which are $\ll x^{3/4}\log x$ by our choice of $V=x^{1/2}$.

The second term in (7) is

$$(2\gamma-1)\,x\sum_{b\le V}\frac{Q_2(b)}{b}=(2\gamma-1)\,x\left(\frac{1}{\zeta(2)}\log V+C_Q+O\Big(\frac{1}{V^{1/2}}\Big)\right),$$

and the error is again $\ll x^{3/4}$.

Finally, the last term in the Hyperbolic Method is

$$\begin{aligned}
\left(\sum_{a\le U}d(a)\right)\left(\sum_{b\le V}Q_2(b)\right) &= \big(U\log U+(2\gamma-1)\,U+O\big(U^{1/2}\big)\big)\\
&\quad\times\left(\frac{1}{\zeta(2)}V+O\big(V^{1/2}\big)\right).
\end{aligned}$$

The error from this is $O\big(U^{1/2}V+UV^{1/2}\log U\big)$ which is $O\big(x^{3/4}\log x\big)$ by the choice of U and V.

Main Terms

The main terms are scattered throughout the above expressions. The main terms will **not** depend on the choice of U and V, the reason for introducing U and V are that good choices for them will give a good bound on the error term.

Terms with two logarithms:

$$
\begin{aligned}
&\frac{x}{\zeta(2)}\frac{1}{2}\log^2 U + x\log\frac{x}{V}\frac{1}{\zeta(2)}\log V + \frac{x}{2\zeta(2)}\log^2 V \\
&\qquad = \frac{x}{2\zeta(2)}\left(\log^2 U + 2\log\frac{x}{V}\log V + \log^2 V\right) \\
&\qquad = \frac{x}{2\zeta(2)}\left(\log^2 U + 2\log U\log V + \log^2 V\right) \qquad\qquad \text{since } UV = x \\
&\qquad = \frac{x}{2\zeta(2)}\left(\log U + \log V\right)^2 = \frac{x}{2\zeta(2)}\left(\log UV\right)^2 \\
&\qquad = \frac{x}{2\zeta(2)}\log^2 x.
\end{aligned}
$$

Terms with one logarithm:

$$
\begin{aligned}
&\frac{x}{\zeta(2)}2\gamma\log U + C_Q x\log\frac{x}{V} + C_Q x\log V \\
&\qquad\qquad + (2\gamma - 1)\frac{x}{\zeta(2)}\log V - (U\log U)\left(\frac{1}{\zeta(2)}V\right) \\
&\quad = \frac{x}{\zeta(2)}2\gamma\left(\log U + \log V\right) + C_Q x\left(\log\frac{x}{V} + \log V\right) - \frac{x}{\zeta(2)}\left(\log V + \log U\right) \\
&\quad = \frac{x}{\zeta(2)}2\gamma\left(\log UV\right) + C_Q x\left(\log\frac{x}{V}V\right) - \frac{x}{\zeta(2)}\left(\log VU\right) \\
&\quad = \left(\frac{(2\gamma-1)}{\zeta(2)} + C_Q\right)x\log x.
\end{aligned}
$$

Terms with no logarithm:

$$
C_d\frac{x}{\zeta(2)} + Dx + (2\gamma-1)\,C_Q x + \frac{(2\gamma-1)}{\zeta(2)}UV = \left(\frac{C_d}{\zeta(2)} + D + (2\gamma-1)\,C_Q + \frac{(2\gamma-1)}{\zeta(2)}\right)x
$$

$\blacksquare$

The Improvement of $\sum_{n\le x} d_3(n)$

If we attempted to improve

$$\sum_{n\le x} d_3(n) = \frac{1}{2} x\log^2 x + O(x\log x)\,,$$

by using the improved estimate for $\sum_{n\le x} d(n)$ from (1) in the Convolution Method, we would get

$$\sum_{n\le x} d_3(n) = \sum_{m\le x}\left(\frac{x}{m}\log\frac{x}{m} + (2\gamma-1)\,\frac{x}{m} + O\!\left(\left(\frac{x}{m}\right)^{1/2}\right)\right).$$

The error term here is

$$\ll x^{1/2}\sum_{m\le x}\frac{1}{m^{1/2}} \ll x.$$

We can get a far smaller error by using the Hyperbolic Method. For this we will need the following result.

Lemma 5 *There exists a constant C_2 such that*

$$\sum_{1\le n\le x}\frac{\log n}{n} = \frac{1}{2}\log^2 x + C_2 + O\!\left(\frac{\log x}{x}\right).$$

Proof Writing the logarithm as an integral an interchanging the summation and integration gives

$$\sum_{n\le x}\frac{\log n}{n} = \sum_{n\le x}\frac{1}{n}\int_1^n\frac{dt}{t} = \int_1^x\frac{dt}{t}\sum_{t<n\le x}\frac{1}{n}.$$

Split this sum as

$$\sum_{t<n\le x}\frac{1}{n} = \sum_{n\le x}\frac{1}{n} - \sum_{n\le t}\frac{1}{n}.$$

Then

$$\begin{aligned}
\int_1^x \frac{dt}{t} \sum_{t<n\le x} \frac{1}{n} &= \left(\sum_{n\le x} \frac{1}{n}\right) \int_1^x \frac{dt}{t} - \int_1^x \sum_{n\le t} \frac{1}{n} \frac{dt}{t} \\
&= \left(\log x + \gamma + O\Big(\frac{1}{x}\Big)\right) \int_1^x \frac{dt}{t} \\
&\qquad - \int_1^x \frac{dt}{t} \big(\log t + \gamma + \varepsilon(t) \big), \qquad \text{where } \varepsilon(t) \ll 1/t, \\
&= \frac{1}{2}\log^2 x + O\Big(\frac{\log x}{x}\Big) - \int_1^\infty \varepsilon(t)\frac{dt}{t} + \int_x^\infty \varepsilon(t)\frac{dt}{t}.
\end{aligned}$$

Since

$$\int_x^\infty \varepsilon(t)\frac{dt}{t} \ll \int_x^\infty \frac{dt}{t^2} \ll \frac{1}{x},$$

the result follows with

$$C_2 = -\int_1^\infty \varepsilon(t)\frac{dt}{t}.$$

■

Lemma 6 *There exists a constant C_d such that*

$$\sum_{n\le X} \frac{d(n)}{n} = \frac{1}{2}\log^2 X + 2\gamma \log X + C_d + O\Big(\frac{1}{X^{1/2}}\Big).$$

Solution Left to student

Theorem 7

$$\sum_{n\le X} d_3(n) = \frac{1}{2} X \log^2 X + AX\log X + BX + O\big(X^{2/3}\log X\big),$$

where $A = 3\gamma - 1$ and $B = C_d + C_2 + 2\gamma^2 - 3\gamma + 1$.

Proof With U and V to be chosen, the Hyperbolic method gives

$$\sum_{n\le X} d_3(n) = \sum_{n\le X} d*1(n) = \sum_{a\le U} d(a) \sum_{b\le X/a} 1 + \sum_{b\le V}\sum_{a\le X/b} d(a) \tag{10}$$

$$- \left(\sum_{a\le U} d(a)\right)\left(\sum_{b\le V} 1\right). \tag{11}$$

The first term on the right hand side equals

$$\begin{aligned}
\sum_{a\le U} d(a)\left[\frac{X}{a}\right] &= \sum_{a\le U} d(a)\left(\frac{X}{a}+O(1)\right)\\
&= X\sum_{a\le U}\frac{d(a)}{a}+O\Bigg(\sum_{a\le U} d\,(a)\Bigg)\\
&= X\left(\frac{1}{2}\log^2 U+2\gamma\log U+C_d+O\Big(\frac{1}{U^{1/2}}\Big)\right) \qquad (12)\\
&\qquad +O\left(U\log U\right),
\end{aligned}$$

having used Lemma 6. The second term on the right hand side of (10) equals

$$\begin{aligned}
\sum_{b\le V}\sum_{a\le X/b} d(a) &= \sum_{b\le V}\left(\frac{X}{b}\log\frac{X}{b}+(2\gamma-1)\,\frac{X}{b}+O\Bigg(\Big(\frac{X}{b}\Big)^{1/2}\Bigg)\right)\\
&\qquad \text{by } (1)\\
&= X\sum_{b\le V}\frac{\log X/b}{b}+(2\gamma-1)\,X\sum_{b\le V}\frac{1}{b}+O\Bigg(X^{1/2}\sum_{b\le V}\frac{1}{b^{1/2}}\Bigg)\\
&= X\left(\log X\sum_{b\le V}\frac{1}{b}-\sum_{b\le V}\frac{\log b}{b}\right)\\
&\qquad +(2\gamma-1)\,X\sum_{b\le V}\frac{1}{b}+O\big(X^{1/2}V^{1/2}\big)\\
&= X\log X\left(\log V+\gamma+O\Big(\frac{1}{V}\Big)\right) \qquad (13)\\
&\qquad -X\left(\frac{1}{2}\log^2 V+C_2+O\Big(\frac{\log V}{V}\Big)\right) \qquad (14)\\
&\qquad +(2\gamma-1)\,X\left(\log V+\gamma+O\Big(\frac{1}{V}\Big)\right) \qquad (15)\\
&\qquad +O\big(X^{1/2}V^{1/2}\big).
\end{aligned}$$

The errors from these two terms in (10) are

$$\ll XU^{-1/2},\ U\log U,\ XV^{-1} \text{ and } X^{1/2}V^{1/2}.$$

We attempt to minimise these errors by choosing U and V to **equalise** them. So try $XV^{-1} = X^{1/2}V^{1/2}$, i.e. $V = X^{1/3}$. Yet $UV = X$ so this means $U = X^{2/3}$. It can be checked that all errors are then $\ll X^{2/3} \log X$.

With this choice of U and V the term in (11) is

$$\left(\sum_{a\leq U} d(a)\right)\left(\sum_{b\leq V} 1\right) = \big(U\log U + (2\gamma-1)\,U + O\big(U^{1/2}\big)\,\big)\,(V + O(1))\,. \quad (16)$$

The error here is $O\big(U\log U + U^{1/2}V\big)$ which is again $\ll X^{2/3}\log X$ given the choice of U and V.

The main terms from (12), (13) and (15) are

$$\begin{aligned} & X\left(\frac{1}{2}\log^2 U + 2\gamma\log U + C_d\right) + X\log X\,(\log V + \gamma) \qquad (17)\\ & -X\left(\frac{1}{2}\log^2 V + C_2\right) + (2\gamma-1)\,X\,(\log V + \gamma)\\ & -V\big(U\log U + (2\gamma-1)\,U\big)\\ = \; & X\left(\frac{1}{2}\log^2 U + \log X\log V - \frac{1}{2}\log^2 V\right)\\ & +X\,(2\gamma\log U + \gamma\log X + (2\gamma-1)\log V - \log U)\\ & +X\big(C_d + C_2 + (2\gamma-1)\,\gamma + (2\gamma-1)\,\big), \end{aligned}$$

having used $UV = X$. Note that these main terms should **not** depend on the choice of U and V, the mean value $\sum_{n\leq X} d_3(n)$ will have the same main terms however they are calculated. What we are doing in this method is not to *calculate exactly* the error for our result on this mean value but to *bound* this error. So a clever choice of U and V will give a better bound *on the error*. To check that the main terms do not depend on the choice of U and

V consider first terms with two logarithms, (dropping the X factor),

$$\begin{aligned}\frac{1}{2}\log^2 U + \log X \log V - \frac{1}{2}\log^2 V &= \frac{1}{2}\log^2 U + \log(UV)\log V - \frac{1}{2}\log^2 V \\ &= \frac{1}{2}\log^2 U + \big(\log U + \log V\big)\log V - \frac{1}{2}\log^2 V \\ &= \frac{1}{2}\log^2 U + \log U \log V + \frac{1}{2}\log^2 V \\ &= \frac{1}{2}\big(\log U + \log V\big)^2 \\ &= \frac{1}{2}\log^2 UV = \frac{1}{2}\log^2 X.\end{aligned}$$

Also, for term with one logarithm (again dropping the X factor)

$$\begin{aligned}&2\gamma \log U + \gamma \log X + (2\gamma - 1)\log V - \log U \\ &\quad = (2\gamma - 1)\big(\log U + \log V\big) + \gamma \log X \\ &\quad = (3\gamma - 1)\log X.\end{aligned}$$

All this combines to give the result quoted. ∎

Note though that now we have justified the claim that it does matter what the choice is for U and V you will always get the same main terms, you can go back to (17) and choose, say, $U = 1$ and $V = X$ when we get

$$\begin{aligned}&= X\left(\frac{1}{2}\log^2 U + \log X \log V - \frac{1}{2}\log^2 V\right) \\ &\quad + X\left(2\gamma \log U + \gamma \log X + (2\gamma - 1)\log V - \log U\right) \\ &\qquad + X\big(C_d + C_2 + (2\gamma - 1)\gamma + (2\gamma - 1)\big) \\ &= X\left(\log X \log X - \frac{1}{2}\log^2 X\right) + X\big(\gamma \log X + (2\gamma - 1)\log X\big) \\ &\quad + X\big(C_d + C_2 + (2\gamma - 1)\gamma + (2\gamma - 1)\big) \\ &= \frac{1}{2}X\log^2 X + (3\gamma - 1)X\log X + X\big(C_d + C_2 + (2\gamma - 1)\gamma + (2\gamma - 1)\big)\end{aligned}$$